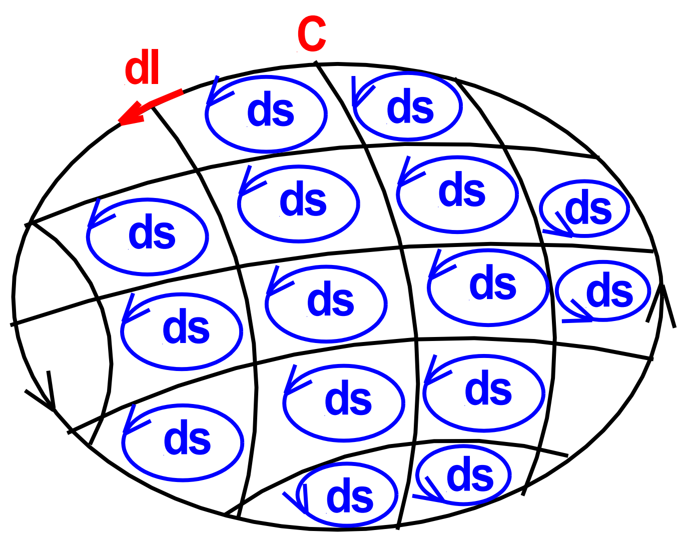
Vector Analysis



www.skywave-radio.org

1 Scalars and Vectors

1.1 Scalar Quantity

A scalar quantity (A) is a quantity that is completely defined by its magnitude and a unit of measure.

1.2 Vector Quantity

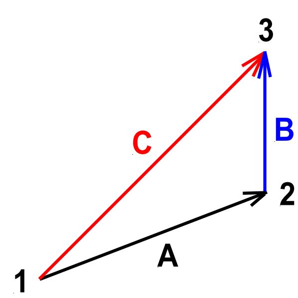
A vector quantity (**A**) is a quantity which is completely defined by its magnitude, direction, and unit of measure. A vector **A** is denoted by making **A** bold. A vector **A** may also be designated by placing an arrow over **A**. That is $\boldsymbol{A} = \overrightarrow{A}$

2 Addition and Subtraction of Vectors

2.1 Addition of Vectors

If vector **A** represents the displacement of a movable object from Point 1 to Point 2, and vector **B** represents the displacement of that same object from Point 2 to Point 3, then the total displacement of the movable object from Point 1 to Point 3 is represented by the resultant vector **C**. The resultant vector **C** is defined as the vector addition of the two vectors **A** and **B**, or

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

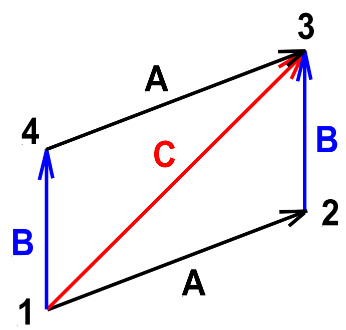


Vector addition satisfies both the Commutative Law and the Associative Law.

Commutative Law: **C** = **A** + **B** = **B** + **A**

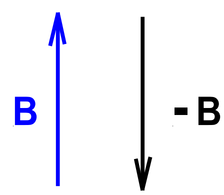
Associative Law: **D** = (**A** + **B**) + **C** = **A** + (**B** + **C**)

It is interesting to note that the Commutative Law of vector addition graphically forms a parallelogram. For this reason, the law of Vector Addition is sometimes referred to as the parallelogram law.



2.2 Subtraction of Vectors

The negative of a vector is a new vector with the same magnitude as the first vector but with opposite direction.



A vector B is subtracted from a vector A by the vector addition of vector A to the negative of vector B

A – **B** = **A** + (-**B**)

3 Multiplying a Vector by a Constant

3.1 Basic Definition

Multiplying a Vector **A** by a positive scalar m results in a new vector **P** in the same direction as **A** but with a magnitude that is m times the magnitude of **A**.

P = m**A**

|**P**| = m|**A**| or

P = mA

where |**P**| = P, and |**A**| = A.

3.2 Unit Vector

A unit vector is a vector whose magnitude is unity. It is often convenient to express a vector as the product of its magnitude and a unit vector having the same direction. If **b** is a unit vector having the same direction as the vector **B**, then

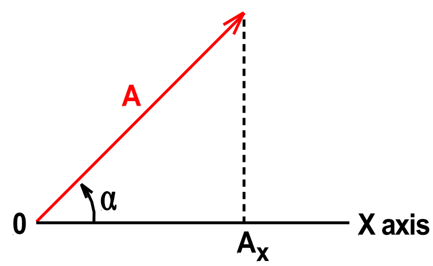
B = B**b**

3.3 Vector Components

If the unit vectors **x**, **y**, and **z** are the unit vectors in the X, Y, and Z directions of a Cartesian coordinate system, then a vector **A** can be represented as:

$$\overrightarrow{A} = A_x\overrightarrow{x} + A_y\overrightarrow{y} + A_z\overrightarrow{z}$$

where A_x is the projection of **A** on the x-axis, A_y is the projection of **A** on the y-axis, and A_z is the projection of **A** on the z-axis.



Projection of **A** on the x-axis

$A_x = |\vec{A}| \cos\alpha$ = Projection of **A** onto the x-axis

$$\frac{A_x}{|\vec{A}|} = \cos\alpha$$

Similarly for the y-axis and the z-axis

$$\frac{A_y}{|\vec{A}|} = \cos\beta$$

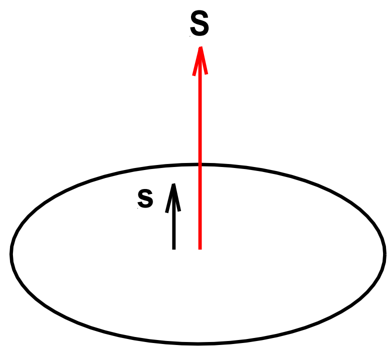
$$\frac{A_z}{|\vec{A}|} = \cos\gamma$$

Finally

$$|\vec{A}|^2 = A_x^2 + A_y^2 + A_z^2$$

4 Vector Representation of a Surface

A surface vector **S** is a vector whose magnitude is equal to the area of the surface and whose direction is perpendicular to the surface, pointed outward for a closed surface, or in accordance with the right hand rule along the surface periphery for an open surface.



A unit surface vector **s** (bold small **s**) is a vector whose magnitude is equal to unity and whose direction is perpendicular to the surface, pointed outward for a closed surface, or in accordance with the right hand rule along the surface periphery for an open surface.

5 The Vector Product of Two Vectors

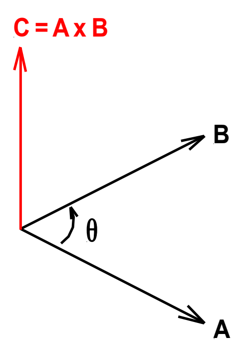
The vector (cross product) **A** x **B** of two vectors **A** and **B** is defined to be a third vector **C**

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \quad \text{(a vector)}$$

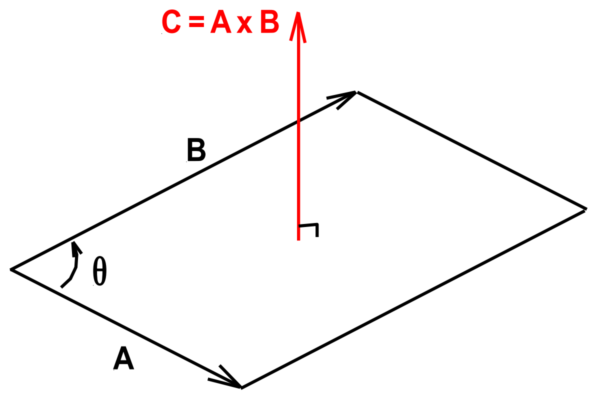
where

$$\text{Magnitude of } C = AB\sin\theta$$

and the direction **C** is normal to the plane formed by **A** and **B** in accordance with the right hand rule, beginning with the first vector (**A**) and rotating through an angle θ to the second vector (**B**).



It is interesting to note that the vector product **C** = **A** x **B** is the surface vector for the parallelogram defined by the two vectors **A** and **B**.



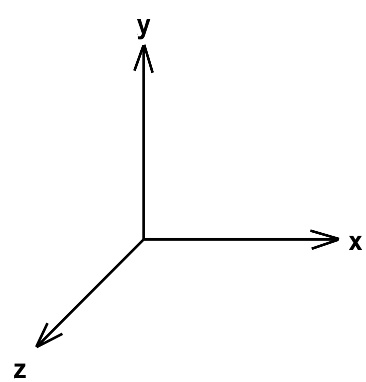
The vector product of **A** x **B** satisfies the distributive law of multiplication,

C x (**A** + **B**) = (**C** x **A**) +(**C** x **B**).

However, since the direction of the resulting vector **C** reverses if the order of multiplication is interchanged, the vector product of **A** x **B** does not satisfy the commutative law.

C = **A** x **B** ≠ **B** x **A**

In terms of unit vectors in the Cartesian coordinate system,



x x **y** = **z**, **y** x **z** = **x**, **z** x **x** = **y**

x x **x** = 0, **y** x **y** = **0**, **z** x **z** = **0**

$$\vec{A} \times \vec{B} = \vec{x}\left(A_y B_z - A_z B_y\right) + \vec{y}(A_z B_x - A_x B_z) + \vec{z}\left(A_x B_y - A_y B_x\right)$$

which is equal to the formal expansion of the following determinant:

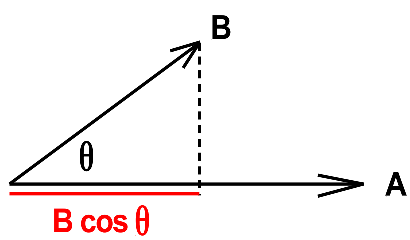
$$\vec{A} \times \vec{B} = \begin{matrix} \vec{x} & \vec{y} & \vec{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{matrix}$$

6 The Scalar (Dot) Product of Two Vectors

The dot product of **A** and **B** is a scalar quantity C whose magnitude is equal to the magnitude of **A** multiplied by the magnitude of the projection of **B** onto **A.**

$$C = \vec{A} \cdot \vec{B} = AB\cos\theta$$

where C is a scalar quantity.



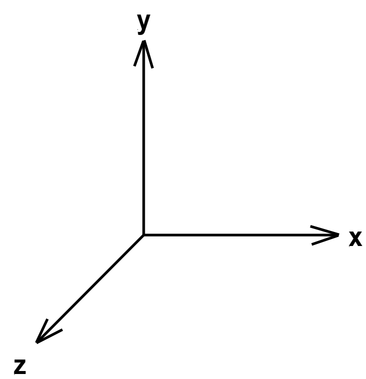
The dot product of **A** and **B** may be interpreted as the magnitude of **A** multiplied by the component of **B** in the direction of **A**.

The dot product of A and B satisfies both the commutative and the distributive laws of multiplication:

Commutative Law: $C = \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

Distributive Law: $D = \left(\vec{A} + \vec{B}\right) \cdot \vec{C} = \left(\vec{A} \cdot \vec{C}\right) + \left(\vec{B} \cdot \vec{C}\right)$

In terms of unit vectors in the Cartesian coordinate system,



$$\vec{x} \cdot \vec{y} = 0 \qquad \vec{y} \cdot \vec{z} = 0 \qquad \vec{z} \cdot \vec{x} = 0$$

$$\vec{x} \cdot \vec{x} = 1 \qquad \vec{y} \cdot \vec{y} = 1 \qquad \vec{z} \cdot \vec{z} = 1$$

7 Scalar and Vector Fields

7.1 General Definition of a Field

In mathematical physics, the field of a physical quantity refers to nothing more than the dependence of that quantity on its position in a region of space. That is, a field exists if that particular physical quantity has a specific value at each point in space. It is assumed that the variation of the physical quantity's value throughout space is, ordinarily, a continuous one. The field may be a scalar field or a vector field.

7.2 A Scalar Field

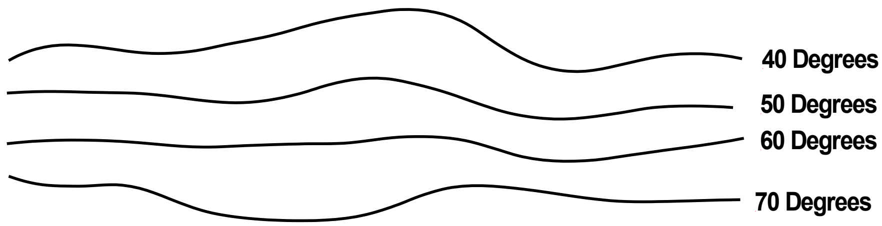
A scalar field is simply the field associated with a scalar physical quantity. A scalar field exists if the physical quantity has a magnitude, but no specific direction, at each point in space. For example, the barometric pressure at each point on the earth's surface constitutes a scalar field. The field is a scalar field because pressure is a scalar quantity.

7.2.1 Proper Scalar Functions

A proper scalar function is a scalar function that varies within a coordinate system but does not depend on the particular coordinate system selected.

7.2.2 Potential Fields

Scalar fields are sometimes called potential fields. Lines or surfaces over which the field has a constant magnitude are referred to as equipotentials.



Equipotential contour lines are used to graphically represent scalar fields.

Isotherms on a weather map are examples of equipotential lines. A temperature can be measured at each point on the surface of the earth. It is convenient to organize this information graphically (or conceptually) by using lines to connect together points that are the same temperature. These lines connecting points of equal temperature are called isotherms on a weather map and give a rough idea of the scalar temperature field.

7.3 Vector Fields

A vector field is the field associated with a vector physical quantity. A vector field exists if the physical quantity has both a magnitude and a direction at each point in space.

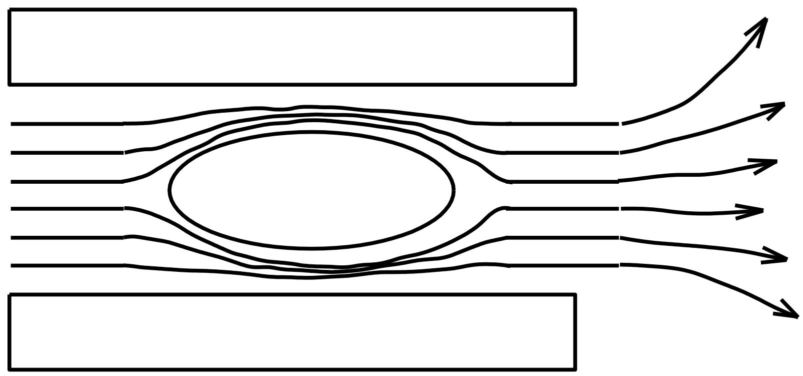
7.3.1 Proper Vector Fields

A vector field may be described in terms of its components at every point in space (x, y, and z components in a cartesian coordinate system). In this way one can form three scalar fields from the vector field. If the component fields are proper scalar functions, then the vector function will be a proper vector function.

Only proper vector functions can enter into physical laws.

7.3.2 Vector Field Flow Lines

Where the vector filed is continuous, we can define conceptual lines of flow for the field. A flow line points in the same direction as the vector field at each point in space. The density of flow lines in a particular region of space represents the magnitude of the vector field in that region. If the density of the flow lines are high, then the magnitude of the vector field in that region is relatively large. If the flow lines are far apart, then the magnitude of the vector field is relatively small.



Flow lines are used to graphically represent vector fields.

The representation of a vector field by means of flow lines seems fairly obvious where the field represents the flow of some quantity. However, the technique of constructing flow lines contains no restriction on such fields. It may be used to represent any vector field. The representation of vector fields through the concept of flow or flux (which means the same thing) is a very powerful visualization tool when working with vector fields.

8 Gradient

Gradient $\overrightarrow{G}$ (a vector quantity) is the direction and magnitude of the maximum rate of change in a scalar field Φ.

$$\overrightarrow{G} = \overrightarrow{a_x}\frac{\partial \Phi}{\partial x} + \overrightarrow{a_y}\frac{\partial \Phi}{\partial y} + \overrightarrow{a_z}\frac{\partial \Phi}{\partial z}$$

The DEL Operator ∇

$$\nabla = \overrightarrow{a_x}\frac{\partial}{\partial x} + \overrightarrow{a_y}\frac{\partial}{\partial y} + \overrightarrow{a_z}\frac{\partial}{\partial z}$$

which is a vector

Gradient $\overrightarrow{G}$ is thus equal to

$$\overrightarrow{G} = \nabla\,\Phi$$

9 Line Integral

If **F** is a vector field and d**l** is a vector element of length along an arbitrary path C, then the integral

$$\int_1^2 \vec{F} \cdot d\vec{l}$$

taken over the extent of the path is called the line integral. Note that the line integral is a scalar quantity. That is

$\int_1^2 \vec{F} \cdot d\vec{l}$ = the line integral of the vector F over the path from Point 1 to Point 2.

A simple problem that leads to a line integral is calculation of the work done in moving a particle along a path C from Point 1 to Point 2, where **F** represents the force applied to the particle at each point along the path C.

The value of the line integral, in general, depends on the path taken to get from Point 1 to Point 2.

However, for certain line integrals, the value depends only on the end points (Point 1 and Point 2) of the path regardless of the actual path taken to get from Point 1 to Point 2.

When the path C is a closed one, the line integral is denoted by:

$$\oint_C \vec{F} \cdot d\vec{l}$$

For a special class of vector fields, **F** can be derived from the gradient of a real, or imagined, scalar field Φ.

The line integral for such a field **F** over a path C from Point 1 to Point 2 can be written as:

$$\int_1^2 \vec{F} \cdot d\vec{l} = \int_1^2 \nabla\Phi \cdot d\vec{l} = \int_1^2 \frac{d\Phi}{dl} dl = \int_1^2 d\Phi = \Phi_2 = \Phi_1$$

Thus the line integral of a vector field **F**, that can be derived from the gradient of a scalar field, depends only on the end points (Point 1 and Point 2) of the path regardless of the actual path taken to get from Point 1 to Point 2.

The line integral of this special class of vector field **F** over a closed path must be zero!

Thus, if the line integral of a vector field **F** around any closed path is zero, the vector field **F** can be derived from the gradient of a scalar field Φ.

10 Surface Integral

10.1 Surface Vector

A surface vector **S** is a vector whose length is equal to the magnitude of the surface area and whose direction is normal to the surface.

When the surface is a closed surface, the positive normal is the direction outward from the closed surface.

For an open surface, the positive normal is the direction specified by the right hand rule with the fingers of the right hand curled around the periphery of the surface and the thumb pointing in the direction of the surface normal.

10.2 The Surface Integral

If **F** is a vector field and d**s** is a vector element of area on a surface S, then the integral

$$\int_S \vec{F} \cdot d\vec{s}$$

taken over the extent of the surface is called the surface integral which is a scalar quantity.

When the surface is a closed surface, the surface integral is denoted by

$$\oint_S \vec{F} \cdot d\vec{s}$$

11 Divergence

The Divergence of a vector field **F** is a scalar quantity that measures the strength of the source at a point which is producing **F**.

div **F** = (Outflow of **F** from the closed surface) / (The volume enclosed by the surface).

$$div\,\vec{F} = \lim_{V \to 0} \frac{\oint_S \vec{F} \cdot d\vec{s}}{V} = \nabla \cdot \vec{F}$$

Thus divergence

$$div\,\vec{F} = \nabla \cdot \vec{F}$$

which is a scalar quantity.

Any type of field, or fluid flow, whether compressible or non-compressible can be represented as a non-compressible field that contains a distribution of sources that create the field and sinks which consume the field.

12 Gauss' Law

The total outflow of a vector field F from a closed surface is equal to the total strength of all the sources, within the volume enclosed by the surface, which are responsible for creating the field.

$$\int_V \left(\nabla \cdot \vec{F}\right) dv = \oint_S \vec{F} \cdot d\vec{s}$$

The total outflow of a field F through a closed surface, $\oint_S \left(\vec{F} \cdot d\vec{s}\right)$, is equal to the divergence of the field integrated throughout the volume enclosed by the surface

$\int_V \left(\nabla \cdot \vec{F}\right) dv$, since the divergence evaluates the net outflow per unit volume.

Johann Carl Friedrich Gauss (30 April 1777 – 23 February 1855) was a German mathematician and physical scientist who contributed significantly to many fields, including number theory, algebra, statistics, analysis, differential geometry, geophysics, electrostatics, astronomy, and optics. Gauss had a remarkable influence in many fields of mathematics and science and is ranked as one of history's most influential mathematicians.

In 1831 Gauss developed a fruitful collaboration with the physics professor Wilhelm Weber, leading to new knowledge in magnetism (including finding a representation for the unit of magnetism in terms of mass, charge, and time) and the discovery of Kirchhoff's circuit laws in electricity. In 1835 he formulated Gauss' law, but it was not published until 1867. It is one of the four Maxwell's equations which form the basis of classical electrodynamics, the other three being Gauss's law for magnetism, Faraday's law of induction, and Ampère's law with Maxwell's correction. Gauss's law can be used to derive Coulomb's law, and vice versa.

13 Curl

Curl is a measure of the magnitude and direction of circulation in a vector field **F**.

The region that produces the circulation is a vortex region, hence the curling of a field may be thought of as the vorticity of the field.

$$curl\,\overrightarrow{F} \equiv \nabla \times \overrightarrow{F}$$

The curl of **F** is a vector quantity. Its component along an arbitrary direction equals the circulation per unit area in the plane normal to that direction.

The vector curl **F** is a measure of a field's vorticity. It corresponds to the maximum circulation per unit area at a point, the maximum being obtained when the area d**s** is so oriented that curl **F** is normal to it.

The curl **F** at a point P is along the axis of rotation of the field close to P.

The curl **F** measures the strength of the **F** field vortex within the closed path.

Curl **F** = The work required to traverse a closed path / The area enclosed by the path.

$$curl\,\overrightarrow{F} = \lim_{s\to 0} \frac{\oint_C \overrightarrow{F} \cdot d\overrightarrow{l}}{S}\vec{s} = \nabla \times \overrightarrow{F}$$

14 Gradient – Divergence – Curl – Laplacian

14.1 Laplacian Operator ∇^2

$$\nabla^2 \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

which is a scalar quantity

14.2 The Curl of a Gradient

$$\nabla \times \nabla \Phi = 0$$

This result is not unexpected since it has already been noted that the gradient is irrotational. Consequently, the curl of the gradient must vanish everywhere.

14.3 Divergence of a Curl

$$\nabla \cdot \nabla \times \vec{F} = 0$$

Any vector field that has zero divergence is called solenoidal field. This describes the fact that the flux lines of such a field are closed on themselves since there are no sources or sinks in the field for the lines of flux to terminate on.

15 Stoke's Theorem

The line integral of a vector field **F** around a closed path C is equal to the curl of the field **F** integrated throughout the surface area enclosed by C.

$$\oint_C \vec{F} \cdot d\vec{l} = \int_S (\nabla \times \vec{F}) \cdot d\vec{s}$$

This is reasonable since the curl evaluates the net circulation per unit area.

Curl F = The work required to traverse a closed path / The area enclosed by the path

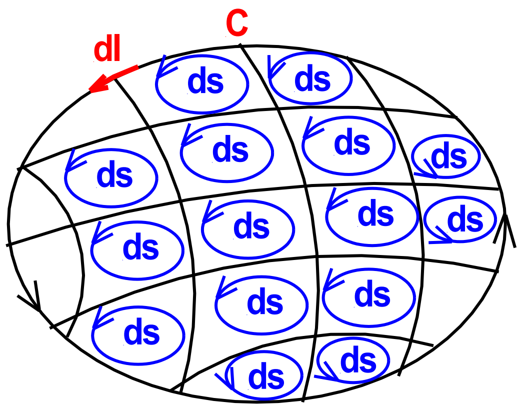
$$curl\,\vec{F} = \lim_{s \to 0} \frac{\oint_C \vec{F} \cdot d\vec{l}}{S} \vec{s} = \nabla \times \vec{F}$$

$$\int_S (\nabla \times \vec{F}) \cdot d\vec{s} = \int_S \lim_{s \to 0} \left[\frac{\oint_C \vec{F} \cdot d\vec{l}}{S} \right] \cdot d\vec{s} = \oint_C \vec{F} \cdot d\vec{l}$$

which is

$$\oint_C \vec{F} \cdot d\vec{l} = \int_S (\nabla \times \vec{F}) \cdot d\vec{s}$$

Conceptually, the "circular" vector in each cell cancels out with the circular vectors in the adjacent cells, except along the circular closed path C.



The above theory was first discovered by Lord Kelvin, who communicated it to George Stokes in a letter dated July 2, 1850. Stokes formalized the theorem in 1854.

16 Helmholtz's Theorem

All vector fields are made up of one or both of the two fundamental types of fields:

Solenoidal Fields: That have zero divergence everywhere, and

Irrotational Fields: That have zero curl everywhere.

Consequentially, a vector field is completely specified by its divergence and curl.

17 Vector Force Field F and Scalar Energy Field Φ

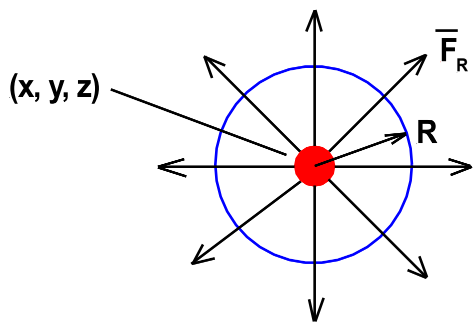
For a single point source located at (x, y, z) symmetry requires that the force field lines **F** be radial and diverge uniformly. If we choose any spherical surface whose center is at the point source, then the total field **F** flowing through the surface will be independent of the sphere radius. In particular, the total field **F** computed is a measure of the total field outflow from the source .... hence it is a measure of the source strength.

If we call the source strength Q and let **F** be the vector force field, then

$$Q = k \oint_S \overrightarrow{F} \cdot d\overrightarrow{s} = k(4\pi R^2)F_R$$

The surface integral is over a spherical surface of radius R, and it can be evaluated because **F** is everywhere radial and of the same magnitude everywhere on the surface S. k is a constant of proportionality to be determined on the basis of our definition of source strength.

$4\pi R^2$ = The surface area of the sphere.



Since

$$Q = k(4\pi R^2)F_R$$

then

$$\overrightarrow{F} = \frac{Q}{k(4\pi R^2)}\overrightarrow{a_R}$$

where $\overrightarrow{a_R}$ is a unit vector in the radial direction.

The vector force field is irrotational, a fact readily established by demonstrating that **F** can be derived as the gradient of a scalar energy field Φ. By inspection it is clear that if

$$\Phi = \frac{Q}{k(4\pi R^2)}$$

then **F** is the negative gradient of Φ, that is

$$\overrightarrow{F} = -\nabla \Phi$$

and

$$\overrightarrow{F} = -\nabla\Phi = \left[\frac{Q}{4k\pi}\overrightarrow{a_R}\right]\frac{\partial\left(\frac{1}{R}\right)}{\partial R} = \frac{-Q}{k(4\pi R^2)}\overrightarrow{a_R}$$

The negative sign is present because

$$\frac{\partial\left(\frac{1}{R}\right)}{\partial R} = \frac{-1}{R^2}$$

References

Plonsey, Robert and Collin, Robert E.; “Principles and Applications of Electromagnetic Fields”, McGraw-Hill Book company, Inc. 1961

Thomas, George B. Jr.; “Calculus and Analytic Geometry” Addison-Wesley Publishing Company, Inc. 1960